Enumerating permutations avoiding more than three Babson - Steingrímsson patterns

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Abstract

Not long ago, Claesson and Mansour proposed some conjectures about the enumeration of the permutations avoiding more than three Babson - Steingrímsson patterns (generalized patterns of type $(1,2)$ or $(2,1)$). The avoidance of one, two or three patterns has already been considered. Here, the cases of four and five forbidden patterns are solved and the exact enumeration of the permutations avoiding them is given, confirming the conjectures of Claesson and Mansour. The approach we use can be easily extended to the cases of more than five forbidden patterns.

1 Introduction

The results of the present paper concern the exact enumeration of the permutations, according to their length, avoiding any set of four or five generalized patterns [BS] of type $(1,2)$ or $(2,1)$. The cases of the permutations avoiding one, two or three generalized patterns (of the same types) were solved in [C], [CM] and [BFP], respectively. In particular, in [CM] the authors conjectured the plausible sequences enumerating the permutations of $S_n(P)$, for any set P of three or more patterns.

In [BFP], the proofs were substantially conducted by finding the ECO construction [BDPP] for the permutations avoiding three generalized patterns of type $(1,2)$ or $(2,1)$, encoding it with a succession rule and, finally, checking that this one leads to the enumerating sequence conjectured in [CM]. This approach could be surely used also for the investigation of the avoidance of four or five generalized patterns of type $(1,2)$ or $(2,1)$ and, maybe, it would allow to find same nice and interesting results: we think that, for instance, in same case new succession rules for known sequences would appear. Nevertheless, this approach has just one obstacle: the large number of cases to consider in order to exhaust all the conjectures in [CM].

The line we are going to follow (see below) is simple and allows us to reduce the number of cases to be considered. Most of the results are summarized in several tables which are presented in Section 4. Really, the paper could appear an easy exercise, but we believe that it is a valuable contribute to the classification of permutations avoiding generalized patterns, started with Claesson, Mansour, Elizalde and Noy [EN], Kitaev [K]. Moreover, it can be seen as the continuation of the work started in [BFP] for the fulfillment of the proofs of the conjectures presented in [CM].

1.1 Preliminaries

A (classical) *pattern* is a permutation $\sigma \in S_k$ and a permutation $\pi \in S_n$ *avoids* σ if there is no any subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ which is order-isomorphic to σ. In other word, π must contain no subsequences having the entries in the same relative order of the entries of σ. *Generalized patterns* were introduced by Babson and Steingrímsson for the study of the mahonian statistics on permutations [BS]. They are constructed by inserting one or more dashes among the elements of a classical pattern (two or more consecutive dashes are not allowed). For instance, $216{-}4{-}53$ is a generalized pattern of length 6. The *type* $(t_1, t_2, \dots, t_{h+1})$ of a generalized pattern containing h dashes records the number of elements between two dashes (we suppose a dash at the beginning and at the end of the generalized pattern, but we omit it): the type of $216-4-53$ is $(3, 1, 2)$. A permutation π *contains* a generalized pattern τ if π contains τ in the classical sense and if any pair of elements of π corresponding to two adjacent elements of τ (not separated by a dash) are adjacent in π, too. For instance, $\pi = 153426$ contains $32-14$ in the entries $\pi_2\pi_3\pi_5\pi_6 = 5326$ or the pattern $3-214$ in the entries $\pi_2\pi_4\pi_5\pi_6 = 5426$. A permutation π *avoids* a generalized pattern τ if it does not contain τ. If P is a set of generalized patterns, we denote $S_n(P)$ the permutations of length n of S (symmetric group) avoiding the patterns of P.

In the paper, we are interested to the generalized patterns of length three, which are of type $(1, 2)$ or $(2, 1)$ and are those ones specified in the set

$$\begin{aligned}\mathcal{M} = \ &\{1-23, 12-3, 1-32, 13-2, 3-12, 31-2, 2-13, 21-3,\\ &\ 2-31, 23-1, 3-21, 32-1\}.\end{aligned}$$

In the sequel, sometimes we can refer to a *generalized pattern of length three* more concisely with *pattern*.

If $\pi \in S$, we define its *reverse* and its *complement* to be the permutations π^r and π^c, respectively, such that $\pi^r_i = \pi_{n+1-i}$ and $\pi^c_i = n + 1 - \pi_i$. We generalize this definition to a generalized pattern τ obtaining its reverse τ^r by reading τ from right to left (regarding the dashes as particular

entries) and its complement τ^c by considering the complement of τ regardless of the dashes which are left unchanged (e.g. if $\tau = 216-4-53$, then $\tau^r = 35-4-612$ and $\tau^c = 561-3-24$). It is easy to check $\tau^{rc} = \tau^{cr}$. If $P \subseteq \mathcal{M}$, the set $\{P, P^r, P^c, P^{rc}\}$ is called the *symmetry class* of P (P^r, P^c and P^{rc} contain the reverses, the complements and the reverse-complements of the patterns specified in P, respectively). We have that $|S_n(P)|$=$|S_n(P^r)|$=$|S_n(P^c)|$=$|S_n(P^{rc})|$ [SS], therefore we can choose one of the four possible sets as the *representative* of a symmetry class, as far as the enumeration of $S(R)$, $R \in \{P, P^r, P^c, P^{rc}\}$, is concerned.

1.2 The strategy

Looking at the table of [CM] where the authors present their conjectures, it is possible to note that most of the sequences enumerating the permutations avoiding four patterns are the same of those ones enumerating the permutations avoiding three patterns. A similar fact happens when the forbidden patterns are four and five. This suggests to use the results for the case of three forbidden patterns (at our disposal) to deduce the proof of the conjectures for the case of four forbidden patterns and, similarly, use the results for the case of four forbidden patterns to solve the case of five forbidden patterns. Indeed, it is obvious that $S(p_1, p_2, p_3, p_4) \subseteq S(p_{i_1}, p_{i_2}, p_{i_3})$ (with $i_j \in \{1,2,3,4\}$ and $p_l \in \mathcal{M}$). If the inverse inclusion can be proved for some patterns, then the classes $S(p_1, p_2, p_3, p_4)$ and $S(p_{i_1}, p_{i_2}, p_{i_3})$ coincide and they are enumerated by the same sequence (a similar argument can be used for the case of four and five forbidden patterns).

The following eight propositions are useful to this aim, as well: each of them proves that if a permutation avoids certain patterns, than it avoids also a further pattern. Therefore, it is possible to apply one of them to a certain class $S(p_{i_1}, p_{i_2}, p_{i_3})$ to prove that $S(p_{i_1}, p_{i_2}, p_{i_3}) \subseteq S(p_1, p_2, p_3, p_4)$ (the generalization to the case of four and five forbidden pattern is straightforward). The proof of the first four of them can be found in [BFP].

Proposition 1.1 *If* $\pi \in S(2-13)$*, then* $\pi \in S(2-13, 21-3)$.

Proposition 1.2 *If* $\pi \in S(31-2)$*, then* $\pi \in S(31-2, 3-12)$.

Proposition 1.3 *If* $\pi \in S(2-31)$*, then* $\pi \in S(2-31, 23-1)$.

Proposition 1.4 *If* $\pi \in S(13-2)$*, then* $\pi \in S(13-2, 1-32)$.

Proposition 1.5 *If* $\pi \in S(1-23, 2-13)$*, then* $\pi \in S(1-23, 2-13, 12-3)$.

Proof. Suppose that π contains a $12-3$ pattern in the entries π_i, π_{i+1} and π_k ($k > i+1$). Let us consider the entry π_{i+2}. It can be neither $\pi_{i+2} > \pi_{i+1}$ (since $\pi_i\ \pi_{i+1}\ \pi_{i+2}$ would show a pattern $1-23$) nor $\pi_{i+2} < \pi_{i+1}$ (since $\pi_{i+1}\ \pi_{i+2}\ \pi_k$ would show a pattern $21-3$ which is forbidden thanks to Proposition 1.1).

$\square$

(The proof of the following proposition is very similar and is omitted.)

Proposition 1.6 *If* $\pi \in S(1-23, 21-3)$, *then* $\pi \in S(1-23, 21-3, 12-3)$.

Proposition 1.7 *If* $\pi \in S(1-23, 2-31)$, *then* $\pi \in S(1-23, 2-31, 12-3)$.

Proof. Suppose that a pattern $12-3$ appear in π_i, π_{i+1} and π_k. If we consider the entry π_{k-1}, then it is easily seen that it can be neither $\pi_i < \pi_{k-1} < \pi_k$ (the entries $\pi_i\ \pi_{k-1}\ \pi_k$ would be $1-23$ pattern like) nor $\pi_{k-1} < \pi_i$ (the entries $\pi_i\ \pi_{i+1}\ \pi_{k-1}$ would show a pattern $23-1$ which is forbidden thanks to Proposition 1.3). Hence, $\pi_{k-1} > \pi_k$. We can repeat the same above argument for the entry π_j, $j = k-2, k-3, \ldots, i+2$, concluding each time that $\pi_j > \pi_{j+1}$. When $j = i+2$ a pattern $1-23$ is shown in $\pi_i\ \pi_{i+1}\ \pi_{i+2}$, which is forbidden.

$\square$

Proposition 1.8 *If* $\pi \in S(1-23, 23-1)$, *then* $\pi \in S(1-23, 23-1, 12-3)$.

This last proposition can be be proved by simply adapting the argument of the proof of the preceding one.

2 Permutations avoiding four patterns

First of all we recall the results of [BFP] in Tables 1 and 2. For the seek of brevity, for each symmetry class only a representative is reported. In the first column of these tables, a name to each symmetry class is given (as in [BFP]), the second one shows the three forbidden patterns (the representative) and the third one indicates the sequence enumerating the permutations avoiding the specified patterns.

Having at our disposal the results for the permutations avoiding three patterns, the proofs for the case of four forbidden patterns are conducted following the line indicated in the previous section. These proofs are all summarized in tables. Tables 3, 4 and 5 are related to the permutations avoiding four patterns enumerated by the sequences $\{n\}_{n\geq 1}$, $\{F_n\}_{n\geq 1}$ and $\{2^{n-1}\}_{n\geq 1}$, respectively (the succession F_n denotes the Fibonacci numbers). As in [BFP], the empty permutation with length $n = 0$ is not considered, therefore the length is $n \geq 1$. The Tables have to be read as follows: consider the representative of the symmetry class specified in the rightmost column of each table; apply the proposition indicated in the precedent column to the three forbidden patterns which one can find in Tables 1 and 2 to obtain the four forbidden patterns written in the column named *avoided patterns*. At this point, as we explained in the previous section, the permutations avoiding

these four patterns are enumerated by the same sequence enumerating the permutations avoiding the three patterns contained in the representative of the symmetry class indicated in the rightmost column.

The first column of Table 3 and 4 specifies a name for the the symmetry class represented by the four forbidden patterns of the second column. This name is useful in the next section. Table 6 indicates in the first column the sequence enumerating the permutations avoiding the patterns of the second column, which are obtained as in the above tables.

2.1 Classes enumerated by $\{0\}_{n\geq k}$.

The classes of four patterns avoiding permutations enumerated by the sequence $\{0\}_{n\geq k}$ can be handled in a very simple way. If $S(q_1, q_2, q_3)$, $q_i \in \mathcal{M}$, is a class of permutations avoiding three patterns such that $|S_n(q_1, q_2, q_3) = 0|$, for $n \geq k$, then it is easily seen that $S(q_1, q_2, q_3, r)$, $\forall r \in \mathcal{M}$, is also enumerated by the same sequence. Then, each symmetry class from $C1$ to $C7$ (see Table 2) generates nine symmetry classes by choosing the pattern $r \neq q_i$, $i = 1, 2, 3$. It is not difficult to see that all the classes we obtain in this way are not all different, thanks to the operations of reverse, complement and reverse-complement. In Table 7, only the different possible cases are presented. Here, the four forbidden patterns are recovered by adding a pattern of a box of the second column to the three patterns specified in the box to its right at the same level (rightmost column). The representative so obtained is recorded in the leftmost column with a name, which will be useful in the next section.

2.2 Classes enumerated by $\{2\}_{n\geq 2}$.

The enumerating sequences encountered till now (see Tables 3, 4, 5, 6, 7) are all involved in the enumeration of some class of permutations avoiding three patterns (Tables 1, 2). Therefore, applying the eight propositions of the previous section to the classes of Table 1 and 2, the three forbidden patterns have been increased by one pattern, obtaining Table 3, 4, 5, 6 and 7. For the classes enumerated by the sequence $\{2\}_{n\geq 2}$ it is not possible to use the same strategy, since there are no classes of permutations avoiding three patterns enumerated by that sequence. The proofs, in this case, use four easy propositions whose proofs can be directly derived from the statement of the first four propositions of the Introduction. We prefer to explicit them the same.

Proposition 2.1 *If a permutation π contains the pattern $23-1$, then it contains the pattern $2-31$, too.*

Taking the reverse, the complement and the reverse-complement of the patterns involved in Prop. 2.1, the following propositions are obtained:

Proposition 2.2 *If a permutation π contains the pattern $1-32$, then it contains the pattern $13-2$, too.*

Proposition 2.3 *If a permutation π contains the pattern $21-3$, then it contains the pattern $2-13$, too.*

Proposition 2.4 *If a permutation π contains the pattern $3-12$, then it contains the pattern $31-2$, too.*

In Table 8 the results relating to the enumeration of the permutations avoiding four patterns enumerated by the sequence $\{2\}_{n\geq 2}$ (whose proofs are contained in the six next propositions) are summarized. The four forbidden patterns can be recovered by choosing one pattern from each column, in the same box-row of the table.

In the sequel, $p_i \in A_i$ with $i = 1, 2, 3, 4$ where A_i is a subset of generalized patterns.

Proposition 2.5 *Let $A_1 = \{1-23\}$, $A_2 = \{2-31, 23-1\}$, $A_3 = \{1-32, 13-2\}$ and $A_4 = \{3-12, 31-2\}$. Then $|S_n(p_1, p_2, p_3, p_4)| = 2$ and $S_n = \{n\ (n-1)\dots 3\ 2\ 1,\ (n-1)\ (n-2)\dots 3\ 2\ 1\ n\}$.*

Proof. Let $\sigma \in S_n(p_2, p_3)$. Then, $\sigma_1 = n$ or $\sigma_n = n$, otherwise, if $\sigma_i = n$ with $i \neq 1, n$, the entries $\sigma_{i-1}\sigma_i\sigma_{i+1}$ would be a forbidden pattern p_2 or p_3.

If $\rho \in S_n(p_1, p_3)$, then $\rho_{n-1} = 1$ or $\rho_n = 1$, otherwise, if $\rho_i = 1$ with $i < n-1$, then the entries $\rho_i\rho_{i+1}\rho_{i+2}$, would be a forbidden pattern p_1 or p_3.

Therefore, if $\pi \in S_n(p_1, p_2, p_3)$, then there are only the following three cases for π:

1. $\pi_n = n$ and $\pi_{n-1} = 1$. In this case $\pi = (n-1)\ (n-2)\dots 2\ 1\ n$, otherwise, if an ascent appears in $\pi_j\pi_{j+1}$ with $j = 1, 2, \dots, n-3$, the entries $\pi_j\pi_{j+1}\pi_{n-1}$ would show the pattern $23-1$ and π would contain the pattern $2-31$, too (see Prop. 2.1).

2. $\pi_1 = n$ and $\pi_n = 1$. In this case $\pi = n\ (n-1)\dots 3\ 2\ 1$, otherwise, if an ascent appears in $\pi_j\pi_{j+1}$ with $j = 2, 3, \dots, n-2$, the entries $\pi_j\pi_{j+1}\pi_n$ would show the pattern $23-1$ and π would contain the pattern $2-31$, too (see Prop. 2.1).

3. $\pi_1 = n$ and $\pi_{n-1} = 1$ (and $\pi_n = k < n$).

If π has to avoid the pattern p_4, too ($\pi \in S_n(p_1, p_2, p_3, p_4)$), then the third above case is not allowed since $\pi_1\pi_{n-1}\pi_n$ are a $3-12$ pattern which induces an occurrence of $31-2$ in π (Prop. 2.4).

□

Proposition 2.6 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{1-32, 13-2\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\dots 3\ 2\ 1,\ (n-1)\ n\ (n-2)\ (n-3)\dots 2\ 1\}$.

Proof. If $\sigma \in S_n(p_1, p_2)$, then $\pi_1 = n$ or $\pi_2 = n$. If $\rho \in S_n(p_1, p_3)$, then $\pi_n = 1$ or $\pi_{n-1} = 1$. Then, if $\pi \in S_n(p_1, p_2, p_3)$, there are only the four following cases:

1. $\pi_1 = n$ and $\pi_n = 1$.

2. $\pi_2 = n$ and $\pi_n = 1$. In this case $\pi_1 = n-1$, otherwise if $\pi_k = n-1$ with $k > 3$, then $\pi_{k-2}\pi_{k-1}\pi_k$ is a $1-23$ pattern or a $21-3$ pattern which induces an occurrence of $2-13$ (Prop. 2.3). If $k = 3$, then $\pi_1\pi_2\pi_3$ is a $1-32$ or $13-2$ pattern which are forbidden.

3. $\pi_1 = n$ and $\pi_{n-1} = 1$.

4. $\pi_2 = n$ and $\pi_{n-1} = 1$. For the same reasons of case 2, it is $\pi_1 = n-1$.

If π has to avoid p_4, too ($\pi \in S_n(p_1, p_2, p_3, p_4)$), then the third and the fourth above cases are not allowed since $\pi_1\pi_{n-1}\pi_n$ are a $3-12$ pattern which induces an occurrence of $31-2$ (Prop. 2.4). Moreover, the permutations of the above cases 1 and 2, must be such that there are not ascents $\pi_i\pi_{i+1}$ between n and 1 in order to avoid p_4. Then, $\pi = n\ (n-1)\dots 3\ 2\ 1$ or $\pi = (n-1)\ n\ (n-2)\dots 3\ 2\ 1$.

□

Proposition 2.7 *Let* $A_1 = \{2-13, 21-3\}$, $A_2 = \{2-31, 23-1\}$, $A_3 = \{1-32, 13-2\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\dots 2\ 1,\ 1\ 2\dots n\}$.

Proof. It is easily seen that each three consecutive elements of π can only be in increasing or decreasing order.

□

Proposition 2.8 *Let* $A_1 = \{12-3\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{2-31, 23-1\}$ *and* $A_4 = \{32-1\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{1\ n\ 2\ (n-1)\dots,\ n\ 1\ (n-1)\ 2\dots\}$.

Proof. If $\pi \in S_n(p_1, p_2, p_3, p_4)$, then it is easy to see that $\pi_1\pi_2 = 1\ n$ or $\pi_1\pi_2 = n\ 1$. Considering the sub-permutation $\pi_2\pi_3\dots\pi_n$, in the same way we deduce $\pi_2\pi_3 = 2\ (n-1)$ or $\pi_2\pi_3 = (n-1)\ 2$. The thesis follows by recursively using the above argument.

□

Proposition 2.9 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{2-31, 23-1\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\ldots 1,\ 1\ n\ (n-1)\ldots 3\ 2\}$.

Proof. Let $\pi \in S_n(p_1, p_2, p_3, p_4)$. It is $\pi_1 = n$ or $\pi_2 = n$, otherwise a $1-23$ or p_2 pattern would appear.

If $\pi_1 = n$, then $\pi = n\ (n-1)\ldots 1$ since if an ascent appears in $\pi_i\pi_{i+1}$, the entries $\pi_1\pi_i\pi_{i+1}$ are a p_4 pattern.

If $\pi_2 = n$, then $\pi_1 = 1$ since the p_3 pattern has to be avoided. Moreover, in this case, it is $\pi_j > \pi_{j+1}$ with $j = 3, 4, \ldots, (n-1)$ in order to avoid $1-23$. Then $\pi = 1\ n\ (n-1)\ldots 2\ 1$.

□

Proposition 2.10 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{2-31, 23-1\}$ *and* $A_4 = \{1-32, 13-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\ldots 3\ 2\ 1,\ n\ (n-1)\ldots 3\ 1\ 2\}$.

Proof. Let $\pi \in S_n(p_1, p_2, p_3, p_4)$. The entries 1 and 2 have to be adjacent in order to avoid p_3 and p_4 and $\pi_n = 1$ or $\pi_{n-1} = 1$ in order to avoid p_1 and p_4. So, $\pi_{n-1}\pi_n = 1\ 2$ or $\pi_{n-1}\pi_n = 2\ 1$. Moreover, each couple of adjacent elements $\pi_j\pi_{j+1}$ must be a descent, otherwise a $23-1$ pattern (which induces an occurrences of $2-31$) would appear. Then $\pi = n\ (n-1)\ldots 3\ 2\ 1$ or $\pi = n\ (n-1)\ldots 3\ 1\ 2$.

□

The conjecture stated in [CM] about the permutations enumerated by $\{2\}_{n\geq 2}$ declares that there are 42 symmetry classes of such permutations, while from Table 8 it is possible to deduce 52 symmetry classes. Nevertheless, it is not difficult to check that these classes are not all different: for example the symmetry class $\{2-13, 2-31, 1-32, 31-2\}$ is the same of $\{2-13, 23-1, 13-2, 31-2\}$ (the second one is the reverse of the first one). Note that the repetitions come out only from the third box-row of Table 8.

3 Permutations avoiding five patterns

3.1 Classes enumerated by $\{1\}_{n\geq 1}$

The sequence $\{1\}_{n\geq 1}$ does not enumerate any class of permutations avoiding four patterns, so that we can not apply the same method of the previous section using the proposition of the Introduction.

Referring to Proposition 2.7, we deduce that there are sixteen different classes $S_n(p_1, p_2, p_3, p_4)$ such that $p_i \in A_i$ with $i = 1, 2, 3, 4$. We recall that $|S_n(p_1, p_2, p_3, p_4)| = 2$ and $S_n(p_1, p_2, p_3, p_4) = \{n\ (n-1)\ldots 2\ 1,\ 1\ 2\ldots n\}$.

If a permutation $\pi \in S_n(p_1, p_2, p_3, p_4)$ has to avoid the pattern $1-23$, too, then $\pi = n\ (n-1) \dots 2\ 1$ and $|S_n(p_1, p_2, p_3, p_4, 1-23)| = 1$.

Then, it is easy to see that the five forbidden patterns avoided by the permutations enumerated by $\{1\}_{n\geq 1}$ can be recovered by considering the four patterns chosen from the third box-row of Table 8 (one pattern from each column) and the pattern $1-23$. We do not present the relative table.

3.2 Classes enumerated by $\{0\}_{n\geq k}$

This case is treated as the case of the permutations avoiding four patterns. It is sufficient to add a pattern $r \in \mathcal{M}$ to each representative (from O1 to O37 in Table 7) of four forbidden patterns of Table 7 in order to obtain a representative T of five forbidden patterns such that $|S_n(T)| = 0, n \geq 4$. In Table 9 we present the different representatives T which can be derived from Table 7. The five forbidden patterns of each representative are a pattern chosen in a box of the first column and the four patterns indicated by the representative (which refer to Table 7) in the second box at the same level. In the table, only the different representatives of five patterns are presented.

3.3 Classes enumerated by $\{2\}_{n\geq k}$, $\{n\}_{n\geq 1}$, $\{F_n\}_{n\geq 1}$

Tables 10 and 11 summarize the results related to the permutations avoiding five patterns enumerated by $\{2\}_{n\geq k}$. The five forbidden patterns are obtained by considering a representative of four forbidden patterns of the rightmost column and the pattern specified in the corresponding box of the preceding column. The first column indicates which is the proposition to apply. Note that each representative of four patterns (rightmost column) can be found in Table 8.

The reading of Tables 12 and 13 (related to the sequences $\{n\}_{n\geq 1}$ and $\{F_n\}_{n\geq 1}$, respectively) is as usual: apply the proposition specified in the first column to recover the representative of five forbidden patterns which is composed by the pattern of the second column and the four patterns of the representative indicated in the rightmost column. Here, the names of the representatives refer to Tables 3 and 4.

4 Tables

symmetry class	avoided patterns	enumerating sequence
N1	{1-23,2-13,3-12}	$\{n\}_{n\geq 1}$
N2	{1-23,2-13,31-2}	
N3	{1-23,21-3,3-12}	
N4	{1-23,21-3,31-2}	
N5	{12-3,3-12,2-13}	
N6	{12-3,3-12,21-3}	
N7	{12-3,31-2,2-13}	
N8	{12-3,31-2,21-3}	
N9	{1-23,2-13,2-31}	
N10	{1-23,2-13,23-1}	
N11	{1-23,21-3,2-31}	
N12	{1-23,21-3,23-1}	
N13	{2-13,2-31,1-32}	
N14	{2-13,23-1,1-32}	
N15	{2-13,2-31,13-2}	
N16	{2-13,23-1,13-2}	
N17	{2-31,21-3,13-2}	
N18	{2-31,21-3,1-32}	
N19	{13-2,21-3,23-1}	
N20	{21-3,23-1,1-32}	
N21	{1-23,2-31,31-2}	
N22	{1-23,23-1,31-2}	
N23	{1-23,2-31,3-12}	
N24	{1-23,1-32,3-21}	
A1	{1-23,12-3,23-1}	$\{2^{n-1}\}_{n\geq 1}$
A2	{2-31,23-1,1-32}	
A3	{2-31,23-1,13-2}	
A4	{1-23,12-3,2-13}	
A5	{1-23,2-13,21-3}	
A6	{1-23,3-12,31-2}	
A7	{31-2,3-12,13-2}	
A8	{31-2,3-12,1-32}	
A9	{2-13,21-3,1-32}	
A10	{2-13,21-3,13-2}	
A11	{1-23,23-1,3-12}	$\{2n-2+1\}_{n\geq 1}$

Table 1: permutations avoiding three patterns

symmetry class	avoided patterns	enumerating sequence
F1	$\{1-23, 2-13, 1-32\}$	$\{F_n\}_{n\geq 1}$
F2	$\{1-23, 2-13, 13-2\}$	
F3	$\{1-23, 21-3, 13-2\}$	
F4	$\{1-23, 13-2, 3-12\}$	
F5	$\{1-23, 1-32, 3-12\}$	
F6	$\{1-23, 1-32, 31-2\}$	
F7	$\{1-23, 13-2, 31-2\}$	
M1	$\{1-23, 12-3, 21-3\}$	$\{M_n\}_{n\geq 1}$
M2	$\{12-3, 21-3, 2-13\}$	
B1	$\{1-23, 21-3, 1-32\}$	$\{\binom{n}{\lceil n/2\rceil}\}_{n\geq 1}$
B2	$\{12-3, 1-23, 31-2\}$	$\{1+\binom{n}{2}\}$
B3	$\{1-23, 2-31, 23-1\}$	
C8	$\{12-3, 2-13, 32-1\}$	$\{3\}_{n\geq 3}$
C1	$\{1-23, 2-13, 3-21\}$	$\{0\}_{n\geq k}$
C2	$\{1-23, 23-1, 32-1\}$	
C3	$\{1-23, 2-13, 32-1\}$	
C4	$\{1-23, 12-3, 3-21\}$	
C5	$\{1-23, 21-3, 3-21\}$	
C6	$\{1-23, 21-3, 32-1\}$	
C7	$\{1-23, 2-31, 32-1\}$	

Table 2: permutations avoiding three patterns

Enumerating sequence: $\{n\}_{n\geq 1}$			
name	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$d1$	$\{1-23, 2-13, 3-12, 21-3\}$	1.1	N1
$d2$	$\{1-23, 2-13, 31-2, 21-3\}$	1.1	N2
$d3$	$\{1-23, 2-13, 31-2, 3-12\}$	1.2	N2
$d4$	$\{1-23, 21-3, 31-2, 3-12\}$	1.2	N4
$d5$	$\{12-3, 3-12, 2-13, 21-3\}$	1.1	N5
$d6$	$\{12-3, 31-2, 2-13, 21-3\}$	1.1	N7
$d7$	$\{12-3, 31-2, 2-13, 3-12\}$	1.2	N7
$d8$	$\{12-3, 31-2, 21-3, 3-12\}$	1.2	N8
$d9$	$\{1-23, 2-13, 2-31, 21-3\}$	1.1	N9
$d10$	$\{1-23, 2-13, 2-31, 23-1\}$	1.3	N9
$d11$	$\{1-23, 2-13, 23-1, 21-3\}$	1.1	N10
$d12$	$\{1-23, 21-3, 2-31, 23-1\}$	1.3	N11
$d13$	$\{2-13, 2-31, 1-32, 21-3\}$	1.1	N13
$d14$	$\{2-13, 2-31, 1-32, 23-1\}$	1.3	N13
$d15$	$\{2-13, 23-1, 1-32, 21-3\}$	1.1	N14
$d16$	$\{2-13, 2-31, 13-2, 21-3\}$	1.1	N15
$d17$	$\{2-13, 2-31, 13-2, 23-1\}$	1.3	N15
$d18$	$\{2-13, 2-31, 13-2, 1-32\}$	1.4	N15
$d19$	$\{2-13, 23-1, 13-2, 21-3\}$	1.1	N16
$d20$	$\{2-13, 23-1, 13-2, 1-32\}$	1.4	N16
$d21$	$\{2-31, 21-3, 13-2, 23-1\}$	1.3	N17
$d22$	$\{2-31, 21-3, 13-2, 1-32\}$	1.4	N17
$d23$	$\{2-31, 21-3, 1-32, 23-1\}$	1.3	N18
$d24$	$\{13-2, 21-3, 23-1, 1-32\}$	1.4	N19
$d25$	$\{1-23, 2-31, 31-2, 23-1\}$	1.3	N21
$d26$	$\{1-23, 2-31, 31-2, 3-12\}$	1.2	N21
$d27$	$\{1-23, 23-1, 31-2, 3-12\}$	1.2	N22
$d28$	$\{1-23, 2-31, 3-12, 23-1\}$	1.3	N23
$d29$	$\{1-23, 2-13, 31-2, 12-3\}$	1.5	N2
$d30$	$\{1-23, 2-13, 3-12, 12-3\}$	1.5	N1
$d31$	$\{1-23, 2-13, 2-31, 12-3\}$	1.5	N9
$d32$	$\{1-23, 2-13, 23-1, 12-3\}$	1.5	N10
$d33$	$\{1-23, 21-3, 2-31, 12-3\}$	1.6	N11
$d34$	$\{1-23, 21-3, 23-1, 12-3\}$	1.6	N12
$d35$	$\{1-23, 21-3, 31-2, 12-3\}$	1.6	N4
$d36$	$\{1-23, 21-3, 3-12, 12-3\}$	1.6	N3
$d37$	$\{1-23, 2-31, 3-12, 12-3\}$	1.7	N23
$d38$	$\{1-23, 2-31, 31-2, 12-3\}$	1.7	N21

Table 3: permutations avoiding four patterns

Enumerating sequence: $\{F_n\}_{n\geq 1}$			
name	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$e1$	$\{1-23, 2-13, 1-32, 21-3\}$	1.1	F1
$e2$	$\{1-23, 2-13, 1-32, 12-3\}$	1.5	F1
$e3$	$\{1-23, 2-13, 13-2, 21-3\}$	1.1	F2
$e4$	$\{1-23, 2-13, 13-2, 1-32\}$	1.4	F2
$e5$	$\{1-23, 2-13, 13-2, 12-3\}$	1.5	F2
$e6$	$\{1-23, 21-3, 13-2, 1-32\}$	1.4	F3
$e7$	$\{1-23, 13-2, 3-12, 1-32\}$	1.4	F4
$e8$	$\{1-23, 1-32, 31-2, 3-12\}$	1.2	F6
$e9$	$\{1-23, 13-2, 31-2, 1-32\}$	1.4	F7
$e10$	$\{1-23, 13-2, 31-2, 3-12\}$	1.2	F7

Table 4: permutations avoiding four patterns

Enumerating sequence: $\{2^{n-1}\}_{n\geq 1}$		
avoided patterns	*apply Proposition*	*to the symmetry class*
$\{1-23, 12-3, 2-13, 21-3\}$	1.1	A4
$\{31-2, 3-12, 13-2, 1-32\}$	1.4	A7
$\{2-13, 21-3, 13-2, 1-32\}$	1.4	A10
$\{2-31, 23-1, 1-32, 13-2\}$	1.4	A3

Table 5: permutations avoiding four patterns

Enumerating sequence	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$\{1+\binom{n}{2}\}_{n\geq 1}$	$\{12-3, 1-23, 31-2, 3-12\}$	1.2	B2
$\{\binom{n}{\lceil n/2\rceil}\}_{n\geq 1}$	$\{1-23, 21-3, 1-32, 12-3\}$	1.6	B1
$\{2^{n-2}+1\}_{n\geq 1}$	$\{1-23, 23-1, 3-12, 12-3\}$	1.8	A11
$\{3\}_{n\geq 3}$	$\{12-3, 2-13, 32-1, 21-3\}$	1.1	C8

Table 6: permutations avoiding four patterns

Enumerating sequence: $\{0\}_{n\geq k}$		
name	*choose a pattern from the following to add*	*to the symmetry class*
O1	$12-3$	$\{1-23, 2-13, 3-21\}$ (C1)
O2	$1-32$	
O3	$13-2$	
O4	$3-12$	
O5	$31-2$	
O6	$21-3$	
O7	$2-31$	
O8	$23-1$	
O9	$32-1$	
O10	$12-3$	$\{1-23, 23-1, 32-1\}$ (C2)
O11	$1-32$	
O12	$13-2$	
O13	$3-12$	
O14	$31-2$	
O15	$2-13$	
O16	$21-3$	
O17	$2-31$	
O18	$3-21$	
O19	$12-3$	$\{1-23, 2-13, 32-1\}$ (C3)
O20	$13-2$	
O21	$3-12$	
O22	$31-2$	
O23	$21-3$	
O24	$2-31$	
O25	$31-2$	$\{1-23, 12-3, 3-21\}$ (C4)
O26	$1-32$	
O27	$23-1$	
O28	$32-1$	
O29	$1-32$	$\{1-23, 21-3, 3-21\}$ (C5)
O30	$13-2$	
O31	$3-12$	
O32	$31-2$	
O33	$23-1$	
O34	$13-2$	$\{1-23, 21-3, 32-1\}$ (C6)
O35	$3-12$	
O36	$2-31$	
O37	$13-2$	$\{1-23, 2-31, 32-1\}$ (C7)

Table 7: permutations avoiding four patterns

Enumerating sequence: $\{2\}_{n\geq 2}$			
1st pattern	*2nd pattern*	*3rd pattern*	*4th pattern*
$1-23$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$1-23$	$2-13$ *or* $21-3$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$12-3$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$32-1$
$1-23$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$3-12$ *or* $31-2$
$1-23$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$

Table 8: permutations avoiding four patterns

Enumerating sequence: $\{0\}_{n\geq k}$	
choose a pattern from the following to add	*to the symmetry class*
$21-3,\ 2-31,\ 23-1,\ 1-32,\ 13-2,$ $3-12,\ 31-2,\ 32-1$	O1
$2-31,\ 23-1,\ 1-32,\ 13-2,\ 3-12,$ $31-2$	O6
$21-3,\ 2-31,\ 23-1,\ 1-32,\ 3-12,$ $31-2$	O19
$2-31,\ 23-1,\ 1-32,\ 13-2,\ 3-12$ $31-2$	O23
$1-32, 13-2, 3-12, 31-2,\ 32-1$	O8
$23-1,\ 1-32,\ 13-2,\ 31-2,\ 3-12$	O24
$12-3,\ 32-1,\ 13-2,\ 3-12,\ 31-2$	O29
$3-12,\ 13-2,\ 1-32,\ 23-1,\ 12-3$	O36
$1-32,\ 13-2,\ 3-12,\ 31-2$	O15
$13-2,\ 3-12,\ 31-2$	O2
$1-32,\ 2-31,\ 31-2$	O10
$12-3,\ 13-2,\ 3-12$	O32
$1-32,\ 13-2,\ 32-1$	O33
$3-21,\ 23-1,\ 1-32$	O34
$3-12,\ 31-2$	O3
$1-32,\ 13-2$	O7
$13-2,\ 3-12$	O9
$21-3,\ 13-2$	O11
$1-32,\ 3-12$	O20
$3-12,\ 23-1$	O26
$2-31,\ 32-1$	O27
$3-12$	O5
$2-31$	O12
$1-32$	O21
$3-12$	O30
$23-1$	O35

Table 9: permutations avoiding five patterns

Enumerating sequence: $\{2\}_{n\geq k}$		
thanks to Proposition	*add the pattern*	*to the patterns*
1.1	$21-3$	$\{1-23, 2-13, 1-32, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$ *or* $\{1-23, 2-13, 13-2, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$
1.1	$21-3$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 1-32\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
1.1	$21-3$	$\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 3-12\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
1.1	$21-3$	$\{1-23, 2-13, 23-1, 32-1\}$ *or* $\{1-23, 2-13, 2-31, 32-1\}$
1.1	$21-3$	$\{2-13, 2-31, 1-32, 3-12\}$ *or* $\{2-13, 2-31, 1-32, 31-2\}$ *or* $\{2-13, 2-31, 13-2, 31-2\}$
1.1	$21-3$	$\{2-13, 23-1, 1-32, 31-2\}$ *or* $\{2-13, 23-1, 1-32, 3-12\}$ *or* $\{2-13, 23-1, 13-2, 31-2\}$ *or* $\{2-13, 23-1, 13-2, 3-12\}$
1.2	$3-12$	$\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
1.2	$3-12$	$\{1-23, 2-13, 1-32, 31-2\}$ *or* $\{1-23, 2-13, 13-2, 31-2\}$
1.2	$3-12$	$\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
1.2	$3-12$	$\{1-23, 21-3, 1-32, 31-2\}$ *or* $\{1-23, 21-3, 13-2, 31-2\}$
1.2	$3-12$	$\{1-23, 2-31, 1-32, 31-2\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$
1.2	$3-12$	$\{1-23, 23-1, 1-32, 31-2\}$ *or* $\{1-23, 23-1, 13-2, 31-2\}$
1.3	$23-1$	$\{2-13, 2-31, 1-32, 31-2\}$
1.3	$23-1$	$\{1-23, 2-31, 13-2, 3-12\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$

Table 10: permutations avoiding five patterns

Enumerating sequence: $\{2\}_{n\geq k}$		
thanks to Proposition	*add the pattern*	*to the patterns*
1.3	$23-1$	$\{1-23, 2-31, 1-32, 3-12\}$ *or* $\{1-23, 2-31, 1-32, 31-2\}$
1.3	$23-1$	$\{1-23, 21-3, 2-31, 1-32\}$ *or* $\{1-23, 21-3, 2-31, 13-2\}$ *or* $\{1-23, 21-3, 2-31, 3-12\}$ *or* $\{1-23, 21-3, 2-31, 31-2\}$
1.3	$23-1$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$
1.4	$1-32$	$\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
1.4	$1-32$	$\{1-23, 2-13, 13-2, 3-12\}$ *or* $\{1-23, 2-13, 13-2, 31-2\}$
1.4	$1-32$	$\{1-23, 21-3, 2-31, 13-2\}$ *or* $\{1-23, 21-3, 23-1, 13-2\}$
1.4	$1-32$	$\{1-23, 21-3, 13-2, 3-12\}$ *or* $\{1-23, 21-3, 13-2, 31-2\}$
1.4	$1-32$	$\{1-23, 2-31, 13-2, 3-12\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$
1.4	$1-32$	$\{1-23, 23-1, 13-2, 3-12\}$ *or* $\{1-23, 23-1, 13-2, 31-2\}$
1.5	$12-3$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 1-32\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
1.5	$12-3$	$\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 3-12\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
1.5	$12-3$	$\{1-23, 2-13, 1-32, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$
1.6	$12-3$	$\{1-23, 21-3, 23-1, 3-12\}$ *or* $\{1-23, 21-3, 23-1, 31-2\}$
1.6	$12-3$	$\{1-23, 21-3, 2-31, 1-32\}$ *or* $\{1-23, 21-3, 23-1, 1-32\}$
1.6	$12-3$	$\{1-23, 21-3, 2-31, 3-12\}$ *or* $\{1-23, 21-3, 2-31, 31-2\}$

Table 11: permutations avoiding five patterns

Enumerating sequence: $\{n\}_{n\geq 1}$		
thanks to Proposition	*add the pattern*	*to the representative*
1.2	$3-12$	d2
1.3	$23-1$	d9
1.2	$3-12$	d6
1.2	$3-12$	d25
1.1	$21-3$	d14
1.1	$21-3$	d17
1.4	$1-32$	d16
1.1	$21-3$	d20
1.3	$23-1$	d18
1.4	$1-32$	d21
1.5	$12-3$	d9
1.5	$12-3$	d11
1.5	$12-3$	d1
1.5	$12-3$	d2
1.5	$12-3$	d10
1.5	$12-3$	d3
1.6	$12-3$	d12
1.6	$12-3$	d4
1.7	$12-3$	d28
1.7	$12-3$	d25

Table 12: permutations avoiding five patterns

Enumerating sequence: $\{F_n\}_{n\geq 1}$		
thanks to Proposition	*add the pattern*	*to the representative*
1.5	$12-3$	e1
1.5	$12-3$	e3
1.1	$21-3$	e4
1.3	$1-32$	e10

Table 13: permutations avoiding five patterns

5 Conclusion: the cases of more than five patterns

The approach we have followed in this work can be used to investigate the enumeration of the permutations avoiding more than five patterns. Really, applying the same propositions (we have herein used) to the results about the case of the avoidance of five patterns, one can try to solve the conjectures

for the case of six patterns. The successive cases can be examined in a similar way.

The case of six patterns is the unique, among the remaining, which presents some enumerating sequence not definitively constant. We note also that all these sequences appear in the enumeration of the case of five patterns. If $|S_n(P)|$ is required, with $P \subseteq \mathcal{M}$, $|P| = 6$, it should take a few minutes to find the set Q of five generalized patterns such that the application of a certain proposition on Q (among those ones presented in this paper) leads to the set P of six forbidden patterns. So $|S_n(Q)| = |S_n(P)|$. Clearly, we are not sure that such a set Q exists since the statements in [CM] are only conjectures. Moreover, it is not sure even the fact that any subset P could be obtained by applying some proposition to some patterns of $Q \subset P$. Nevertheless, the application of the above mentioned propositions to the sets Q of five forbidden patterns should be confirm most of the conjectures about the case of six patterns. This is the reason why we did not present the analysis of this case, together with the fact that several other tables would have appeared in these pages.

To conclude, we think that a further work about the cases of more than six forbidden patterns does not seem to be necessary, since many of the remaining conjectures in [CM] can be easily proved. Moreover, if $S_n(P)$ is needed, with $|P| > 6$, an argument similar to the case $|P| = 6$ can be done.

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